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Original Article

Orlicz difference sequence spaces generated by infinite matrices and de la Vallée-Poussin mean of order α

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Abstract In this paper we introduce the spaces $\widehat{V}_\lambda[A, M, \Delta, p]_o$, $\widehat{V}_\lambda[A, M, \Delta, p]$ and $\widehat{V}_\lambda[A, M, \Delta, p]_\infty$ generated by infinite matrices defined by Orlicz functions. Also we introduce the concept of $\widehat{S}_\lambda[A, \Delta]-$convergence and derive some results between the spaces $\widehat{S}_\lambda[A, \Delta]$ and $\widehat{V}_\lambda[A, \Delta]$. Further, we study some geometrical properties such as order continuity, the Fatou property and the Banach–Saks property of the new space $\widehat{V}^\alpha_\lambda[A, \Delta, p]_\infty$. Finally, we introduce the notion of almost λ-statistically-$[A, \Delta]$-convergence of order α or $\widehat{S}^\alpha_\lambda[A, \Delta]-$convergence and obtain some inclusion relations between the set $\widehat{S}^\alpha_\lambda[A, \Delta]$ and the space $\widehat{V}^\alpha_\lambda[A, \Delta, p]_\infty$.

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1. Introduction

We denote w, ℓ_∞, c and c_0, the spaces of all, bounded, convergent, null sequences, respectively. Also, by ℓ_1 and ℓ_p, we denote the spaces of all absolutely summable and p-absolutely summable series, respectively. Also we denote c_{00} the space of real sequences which have only a finite number of non-zero coordinates. Recall that a sequence $(x(i))_{i=1}^\infty$ in a Banach space X is called *Schauder* (or *basis*) of X if for each $x \in X$ there exists a unique sequence $(a(i))_{i=1}^\infty$ of scalars such that

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$x = \sum_{i=1}^{\infty} a(i)x(i)$, i.e. $\lim_{n\to\infty} \sum_{i=1}^{n} a(i)x(i) = x$. A sequence space X with a linear topology is called a *K-space* if each of the projection maps $P_i : X \to \mathbb{C}$ defined by $P_i(x) = x(i)$ for $x = (x(i))_{i=1}^{\infty} \in X$ is continuous for each natural i. A *Fréchet space* is a complete metric linear space and the metric is generated by a *F-norm* and a Fréchet space which is a *K-space* is called an *FK-space* i.e. a *K-space* X is called an *FK-space* if X is a complete linear metric space. In other words, X is an *FK-space* if X is a Fréchet space with continuous coordinatewise projections. All the sequence spaces mentioned above are *FK*-space except the space c_{00}. An *FK*-spaces X which contains the space c_{00} is said to have the *property AK* if for every sequence $(x(i))_{i=1}^{\infty} \in X$, $x = \sum_{i=1}^{\infty} x(i)e(i)$ where $e(i) = (0, 0, \ldots 1^{i\text{th}place}, 0, 0, \ldots)$.

A Banach space X is said to be a *Köthe sequence space* if X is a subspace of w such that

(a) if $x \in w$, $y \in X$ and $|x(i)| \le |y(i)|$ for all $i \in \mathbb{N}$, then $x \in X$ and $\|x\| \le \|y\|$
(b) there exists an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbb{N}$.

We say that $x \in X$ is *order continuous* if for any sequence $(x_n) \in X$ such that $x_n(i) \le |x(i)|$ for all $i \in \mathbb{N}$ and $x_n(i) \to 0$ as $n \to \infty$ we have $\|x_n\| \to 0$ as $n \to \infty$ holds.

A Köthe sequence space X is said to be *order continuous* if all sequences in X are order continuous. It is easy to see that $x \in X$ order continuous if and only if $||(0, 0, \ldots, 0, x(n+1), x(n+2), \ldots)|| \to 0$ as $n \to \infty$.

A Köthe sequence space X is said to have the *Fatou property* if for any real sequence x and (x_n) in X such that $x_n \uparrow x$ coordinatewisely and $\sup_n ||x_n|| < \infty$, we have that $x \in X$ and $\|x_n\| \to \|x\|$ as $n \to \infty$.

A Banach space X is said to have the *Banach–Saks property* if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $(t_k(z))$ is convergent in X with respect to the norm, where

$$t_k(z) = \frac{z_1 + z_2 + \cdots + z_k}{k} \text{ for all } k \in \mathbb{N}.$$

Some of works on geometric properties of sequence space can be found in [1–4].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \ge 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$,
4. if (γ_k) is a sequence of scalars with $\gamma_k \to \gamma$, as $k \to \infty$ and (x_k) is a sequence of vectors with $p(x_k - x) \to 0$ as $k \to \infty$, then $p(\gamma_k x_k - \gamma x) \to 0$ as $k \to \infty$.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k p_k < \infty$, then for any complex numbers a_k and b_k

$$|a_k + b_k|^{p_k} \le C\big(|a_k|^{p_k} + |b_k|^{p_k}\big) \tag{1.1}$$

where $C = \max(1, 2^{H-1})$. Also, for any complex number α, (see [5])

$$|\alpha|^{p_k} \le \max\big(1, |\alpha|^{H}\big). \tag{1.2}$$

A function M: $[0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if it is continuous, convex, nondecreasing function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \le M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [6]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$.

Lemma 1.1. *An Orlicz function satisfies the inequality* $M(\lambda x) \le \lambda M(x)$ *for all* λ *with* $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{(x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0\right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \le 1\right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \le p < \infty$.

2. Classes of Orlicz difference sequences

The strongly almost summable sequence spaces were introduced and studied by Maddox [5], Nanda [8], Güngör et al., [9], Esi [10], Güngör and Et [11], Esi and Et [12] and many authors.

Let $\lambda = (\lambda_r)$ be a monotonically increasing sequence of positive real numbers tending to ∞ such that $\lambda_r \le \lambda_r + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by $t_r(x) = \frac{1}{\lambda_r} \sum_{k\in I_r} x_k$ where $I_r = [r - \lambda_r + 1, r]$ for $r = 1, 2, 3, \ldots$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ (see [13]). If $\lambda_r = r$, then (V, λ)-summability is reduced to Cesáro summability. We denote Λ the set of all increasing sequences of positive real numbers tending to ∞ such that $\lambda_r \le \lambda_r + 1$, $\lambda_1 = 1$.

Let A= (a_{ij}) be an infinite matrix of non-negative real numbers with all rows are linearly independent for all $i, j = 1, 2, 3, \ldots$ and $B_{kn}(x) = \sum_{i=1}^{\infty} a_{ki}x_{n+i}$ and, the series $\sum_{i=1}^{\infty} a_{ki}x_{n+i}$ converges for each k and uniformly on n.

Let M be an Orlicz function, $p = (p_k)$ be a sequence of positive real numbers, and $\lambda = (\lambda_r)$ be a monotonically increasing sequences of positive real numbers. For $\rho > 0$ we define the new sequence spaces as follows:

$$\widehat{V}_\lambda[A, M, \Delta, p]_o = \left\{x \in w : \lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k} = 0, \text{ uniformly on } n\right\},$$

$$\widehat{V}_\lambda[A, M, \Delta, p] = \left\{x \in w : \lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} = 0, \text{ for some } L, \text{ uniformly on } n\right\}$$

and

$$\widehat{V}_\lambda[A, M, \Delta, p]_\infty = \left\{ x \in w : \sup_r \frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} < \infty, \text{ uniformly on } n \right\},$$

where $\Delta B_{kn}(x) = \sum_{i=1}^{\infty} \left(a_{ki} - a_{k+1,i}\right) x_{n+i}$.

Theorem 2.1. *For an Orlicz function M and a bounded sequence $p = (p_k)$ of positive real numbers, $\widehat{V}_\lambda[A, M, \Delta, p]_o$, $\widehat{V}_\lambda[A, M, \Delta, p]$ and $\widehat{V}_\lambda[A, M, \Delta, p]_\infty$ are linear spaces over the set of complex field.*

Proof. We give the proof only for the space $\widehat{V}_\lambda[A, M, \Delta, p]_o$ and for other spaces follow by applying similar method. Let $x = (x_k), y = (y_k) \in \widehat{V}_\lambda[A, M, \Delta, p]_o$ and $\alpha, \beta \in \mathbb{C}$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1} \right) \right]^{p_k} = 0 \text{ uniformly on } n$$

and

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2} \right) \right]^{p_k} = 0 \text{ uniformly on } n.$$

Define $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since the operator ΔB_{kn} is linear and M is non-decreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(\alpha x + \beta y)|}{\rho_3} \right) \right]^{p_k} \\
&= \frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\alpha \Delta B_{kn}(x) + \beta \Delta B_{kn}(y)|}{\rho_3} \right) \right]^{p_k} \\
&\le \frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\alpha \Delta B_{kn}(x)|}{\rho_3} \right) + M\left(\frac{|\beta \Delta B_{kn}(y)|}{\rho_3} \right) \right]^{p_k} \\
&\le \frac{1}{\lambda_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1} \right) + M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2} \right) \right]^{p_k} \\
&\le \frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1} \right) + M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2} \right) \right]^{p_k} \\
&\le \frac{C}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1} \right) \right]^{p_k} + \frac{C}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2} \right) \right]^{p_k} \\
&\to 0 \text{ as } r \to \infty
\end{aligned}$$

where $C = \max(1, 2^{H-1})$, so $\alpha x + \beta y \in \widehat{V}_\lambda[A, M, \Delta, p]_o$, hence it is a linear space. $\square$

Theorem 2.2. *For an Orlicz function M and a bounded sequence $p = (p_k)$ of positive real numbers, $\widehat{V}_\lambda[A, M, \Delta, p]_o$ is a topological linear space, paranormed by*

$$g(x) = \inf \left\{ \rho^{\frac{p_r}{H}} : \left(\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{T}} \le 1, \ r = 1, 2, 3, \ldots \right\}$$

where $T = \max(1, \sup_k p_k = H)$.

Proof. The subadditivity of g follows from the Theorem 2.1, by taking $\alpha = \beta = 1$ and it is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get $\inf\{\rho^{\frac{p_r}{H}}\} = 0$ for $x = 0$. Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Delta B_{kn}(x) \neq 0$ for each k and uniformly on n. Let $\varepsilon \to 0$, then

$$\frac{|\Delta B_{kn}(x)|}{\varepsilon} \to \infty.$$

It follows that

$$\left(\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\varepsilon} \right) \right]^{p_k} \right)^{\frac{1}{T}} \to \infty$$

which is a contradiction.

Next we prove that scalar multiplication is continuous. Let γ be any complex number, by definition

$$\begin{aligned}
g(\gamma x) &= \inf \left\{ \rho^{\frac{p_r}{H}} : \left(\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(\gamma x)|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{T}} \le 1, \ r = 1, 2, 3, \ldots \right\} \\
&= \inf \left\{ \rho^{\frac{p_r}{H}} : \left(\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\gamma||\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{T}} \le 1, \ r = 1, 2, 3, \ldots \right\}.
\end{aligned}$$

Suppose that $s = \frac{\rho}{|\gamma|}$, then $\rho = s|\gamma|$ and since $|\gamma|^{p_k} \le \max\left(1, |\gamma|^H\right)$ we have

$$g(\gamma x) \le |\gamma|^{p_k} \le \max\left(1, |\gamma|^H\right) \inf \times \left\{ s^{\frac{p_r}{H}} : \left(\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{s} \right) \right]^{p_k} \right)^{\frac{1}{T}} \le 1, \ r = 1, 2, 3, \ldots \right\}$$

which converges to zero as x converges to zero in $\widehat{V}_\lambda[A, M, \Delta, p]_o$. Now suppose that $\lambda_i \to \infty$ as $i \to \infty$ and x is fixed in $\widehat{V}_\lambda[A, M, \Delta, p]_o$. For arbitrary $\varepsilon > 0$ and let r_o be a positive integer such that

$$\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} \le \left(\frac{\varepsilon}{2} \right)^T$$

for some $\rho > 0$ and $r > r_o$. This implies that

$$\left(\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\gamma \Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{T}} < \frac{\varepsilon}{2}$$

for some $\rho > 0$ and $r > r_o$. Let $0 < |\gamma| < 1$. Using the convexity of Orlicz function M, for $r > r_o$, we get

$$\frac{1}{\lambda_r} \sum_{k \in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k}$$

$$\leq \frac{1}{\lambda_r}\sum_{k\in I_r}\left[M\left(\frac{|\gamma||\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^T.$$

Since M is continuous everywhere in $[0, \infty)$, then we consider for $r > r_o$ the function

$$f(t) = \frac{1}{\lambda_r}\sum_{k\in I_r}\left[M\left(\frac{|t\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}.$$

Then f is continuous at zero. So there is a $\delta \in (0, 1)$ such that $|f(t)| < \left(\frac{\varepsilon}{2}\right)^T$ for $0 < t < \delta$. Therefore

$$\left(\frac{1}{\lambda_r}\sum_{k\in I_r}\left[M\left(\frac{|\gamma\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{T}} < \frac{\varepsilon}{2},$$

so that $g(\gamma x) \to 0$ as $\gamma \to 0$. This completes the proof. □

Theorem 2.3. *Let the sequence $p = (p_k)$ be bounded. Then $\widehat{V}_\lambda[A, M, \Delta, p]_o \subset \widehat{V}_\lambda[A, M, \Delta, p] \subset \widehat{V}_\lambda[A, M, \Delta, p]_\infty$.*

Proof. Let $x = (x_k) \in \widehat{V}_\lambda[A, M, \Delta, p]_o$. Then we have

$$\begin{aligned}
&\frac{1}{\lambda_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{2\rho}\right)\right]^{p_k}\\
&\quad\leq \frac{C}{\lambda_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k}\\
&\qquad+\frac{C}{\lambda_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|L|}{\rho}\right)\right]^{p_k}\\
&\quad\leq \frac{C}{\lambda_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k}\\
&\qquad+C\max\left(1, \sup\left[M\left(\frac{|L|}{\rho}\right)\right]^H\right),
\end{aligned}$$

where $H = \sup_k p_k < \infty$ and $C = \max\left(1, 2^{H-1}\right)$. Thus we have $x = (x_k) \in \widehat{V}_\lambda[A, M, \Delta, p]$. The inclusion $\widehat{V}_\lambda[A, M, \Delta, p] \subset \widehat{V}_\lambda[A, M, \Delta, p]_\infty$ is obvious. □

3. New set of sequences of order α

In this section let $\alpha \in (0, 1]$ be any real number, let $\lambda = (\lambda_r)$ be a monotonically increasing sequence of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_r + 1$, $\lambda_1 = 1$, and p be a positive real number such that $1 \leq p < \infty$.

Now we define the following sequence space.

$$\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) = \left\{x \in w : \sup_r \frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p < \infty, \text{ uniformly on } n.\right\}$$

Special cases:

(a) For $p = 1$ we have $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) = \widehat{V}_\lambda^\alpha[A, \Delta]_\infty$.
(b) For $\alpha = 1$ and $p = 1$ we have $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) = \widehat{V}_\lambda[A, \Delta]_\infty$.

Theorem 3.1. *Let $\alpha \in (0, 1]$ and p be a positive real number such that $1 \leq p < \infty$. Then the sequence space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ is a BK-space normed by*

$$||x||_\alpha = \sup_r \frac{1}{\lambda_r^\alpha}\left(\sum_{k\in I_r}|\Delta B_{kn}(x)|^p\right)^{\frac{1}{p}}.$$

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.2. *Let $\alpha \in (0, 1]$ and p be a positive real number such that $1 \leq p < \infty$. Then $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty \subset \widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$.*

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.3. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. Then $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) \subset \widehat{V}_\lambda^\beta[A, \Delta]_\infty(p)$.*

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.4. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. For any two sequences $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ for all r, then $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) \subset \widehat{V}_\mu^\beta[A, \Delta]_\infty(p)$ if and only if $\sup_r(\frac{\lambda_r^\alpha}{\mu_r^\beta}) < \infty$.*

Proof. Let $x = (x_k) \in \widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ and $\sup_r(\frac{\lambda_r^\alpha}{\mu_r^\beta}) < \infty$. Then

$$\sup_r \frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p < \infty$$

and there exists a positive number K such that $\lambda_r^\alpha \leq K\mu_r^\beta$ and so that $\frac{1}{\mu_r^\beta} \leq \frac{K}{\lambda_r^\alpha}$ for all r. Therefore, we have

$$\frac{1}{\mu_r^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p \leq \frac{K}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p.$$

Now taking supremum over r, we get

$$\sup_r \frac{1}{\mu_r^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p \leq \sup_r \frac{K}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p$$

and hence $x \in \widehat{V}_\mu^\beta[A, \Delta]_\infty(p)$.

Next suppose that $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) \subset \widehat{V}_\mu^\alpha[A, \Delta]_\infty(p)$ and $\sup_r(\frac{\lambda_r^\alpha}{\mu_r^\beta}) = \infty$. Then there exists an increasing sequence (r_i) of natural numbers such that $\lim_i(\frac{\lambda_{r_i}^\alpha}{\mu_{r_i}^\beta}) = \infty$. Let L be a positive real number, then there exists $i_0 \in \mathbb{N}$ such that $\frac{\lambda_{r_i}^\alpha}{\mu_{r_i}^\beta} > L$ for all $r_i \geq i_0$. Then $\lambda_{r_i}^\alpha > L\mu_{r_i}^\beta$ and so $\frac{1}{\mu_{r_i}^\beta} > \frac{L}{\lambda_{r_i}^\alpha}$. Therefore we can write

$$\frac{1}{\mu_{r_i}^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p > \frac{L}{\lambda_{r_i}^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p \text{ for all } r_i \geq i_0.$$

Now taking supremum over $r_i \geq i_0$ then we get

$$\sup_{r_i\geq i_0}\frac{1}{\mu_{r_i}^\beta}\sum_{k\in I_{r_i}}|\Delta B_{kn}(x)|^p > \sup_{r_i\geq i_0}\frac{L}{\lambda_{r_i}^\alpha}\sum_{k\in I_{r_i}}|\Delta B_{kn}(x)|^p. \tag{3.1}$$

Since the relation (3.1) holds for all $L \in \mathbb{R}^+$ (we may take the number L sufficiently large), we have

$$\sup_{r_i \geq i_0} \frac{1}{\mu_{r_i}^\beta} \sum_{k \in I_{r_i}} |\Delta B_{kn}(x)|^p = \infty$$

but $x = (x_k) \in \widehat{V}_\lambda^\alpha[A, \Delta, p]_\infty$ with

$$\sup_r \left(\frac{\lambda_r^\alpha}{\mu_r^\beta} \right) < \infty.$$

Therefore $x \notin \widehat{V}_\mu^\alpha[A, \Delta]_\infty(p)$ which contradicts that $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) \subset \widehat{V}_\mu^\alpha[A, \Delta]_\infty(p)$. Hence $\sup_{r \geq 1} \left(\frac{\lambda_r^\alpha}{\mu_r^\beta} \right) < \infty$. □

Corollary 3.5. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. Then for any two sequences $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ for all $r \geq 1$*

(a) $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) = \widehat{V}_\mu^\beta[A, \Delta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{\lambda_r^\alpha}{\mu_r^\beta} \right) < \sup_r \left(\frac{\lambda_r^\alpha}{\mu_r^\beta} \right) < \infty$.

(b) $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) = \widehat{V}_\mu^\alpha[A, \Delta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{\lambda_r^\alpha}{\mu_r^\alpha} \right) < \sup_r \left(\frac{\lambda_r^\alpha}{\mu_r^\alpha} \right) < \infty$.

(c) $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) = \widehat{V}_\lambda^\beta[A, \Delta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{\lambda_r^\alpha}{\lambda_r^\beta} \right) < \sup_r \left(\frac{\lambda_r^\alpha}{\lambda_r^\beta} \right) < \infty$.

Theorem 3.6. $\ell_p[A, \Delta] \subset \widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) \subset \ell_\infty[A, \Delta]$.

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.7. *If $0 < p < q$, then $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p) \subset \widehat{V}_\lambda^\alpha[A, \Delta]_\infty(q)$.*

Proof. The proof of the result is straightforward, so omitted. □

4. Some geometric properties of the new space

In this section we study some of the geometric properties like order continuity, the Fatou property and the Banach–Saks property of type p in this new sequence space.

Theorem 4.1. *The space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ is order continuous.*

Proof. To show that the space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ is an AK-space. It is easy to see that $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ contains c_{00}. By using the definition of AK-properties, we have that $x = (x(i)) \in \widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ has a unique representation $x = \sum_{i=1}^\infty x(i)e(i)$ i.e. $||x - x^{[j]}||_\alpha = ||(0, 0, \ldots, x(j), x(j+1), \ldots)||_\alpha \to 0$ as $j \to \infty$, which means that $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ has AK. Therefore FK-space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ contains c_{00} has AK-property. Also since $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ is a Köthe space, hence the space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ is order continuous. □

Theorem 4.2. *The space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ has the Fatou property.*

Proof. Let x be a real sequence and (x_j) be any nondecreasing sequence of non-negative elements from $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ such that $x_j(i) \to x(i)$ as $j \to \infty$ coordinatewisely and $\sup_j ||x_j||_\alpha < \infty$.

Let us denote $T = \sup_j ||x_j||_\alpha$. Since the supremum is homogeneous, then we have

$$\frac{1}{T} \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} |\Delta B_{kn}(x_j(i))|^p \right)^{\frac{1}{p}} \leq \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} \left| \frac{\Delta B_{kn}(x_j(i))}{||x_n||_\alpha} \right|^p \right)^{\frac{1}{p}}$$
$$= \frac{1}{||x_n||_\alpha} ||x_n||_\alpha = 1.$$

Also by the assumptions that (x_j) is non-decreasing and convergent to x coordinatewisely and by the Beppo-Levi theorem, we have

$$\frac{1}{T} \lim_{j \to \infty} \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} |\Delta B_{kn}(x_j(i))|^p \right)^{\frac{1}{p}}$$
$$= \sup_r \frac{1}{\lambda_r^\alpha} \left(\sum_{k \in I_r} \left| \frac{\Delta B_{kn}(x(i))}{T} \right|^p \right)^{\frac{1}{p}} \leq 1,$$

whence

$$||x||_\alpha \leq T = \sup_j ||x_j||_\alpha = \lim_{j \to \infty} ||x_j||_\alpha < \infty.$$

Therefore $x \in \widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$. On the other hand, for any natural number j the sequence (x_j) is non-decreasing, we obtain that the sequence $(\|x_j\|_\alpha)$ is bounded form above by $\|x\|_\alpha$. Therefore $\lim_{j \to \infty} \|x_j\|_\alpha \leq \|x\|_\alpha$ which contradicts the above inequality proved already, yields that $||x||_\alpha = \lim_{j \to \infty} ||x_j||_\alpha$. □

Theorem 4.3. *The space $\widehat{V}_\lambda^\alpha[A, \Delta]_\infty(p)$ has the Banach–Saks property.*

Proof. The proof of the result follows from the used in [1]. □

5. λ-statistical convergence

The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition Zygmund [14]. Later, this idea was introduced by Fast [15] and Steinhaus [16] and studied various authors (see [10,17,18]). Mursaleen [19], introduced the notion λ-statistical convergence for real sequences. For more details on λ-statistical convergence we refer to [20] and many others. The notion of order statistical convergence was introduced by Gadjiev and Orhan [21] and after that statistical convergence of order α studied by Çolak [22], λ-statistical convergence of order α studied by Çolak and Bektaş [23], λ-statistical convergence of order α of sequence of functions studied by Et et al., [24,25] and many authors. In this section, we define the concept of $\widehat{S}_\lambda[A, \Delta]$-convergence and establish the relationship of $\widehat{S}_\lambda[A, \Delta]$ with $\widehat{V}_\lambda[A, \Delta]$. Also we introduce the notion of $\widehat{S}_\lambda[A, \Delta]$−convergence of order α of real number sequences and obtain some inclusion relations between the set of $\widehat{S}[A, \Delta]$−convergence of order α and the sets $\hat{V}_\lambda^\alpha[A, \Delta]$ and $\hat{V}_\lambda^\alpha[A, M, \Delta, p]$.

Definition 5.1. [19] A sequence $x = (x_k)$ is said to be λ-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{\lambda_r} |\{k \in I_r : \ |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$.

Definition 5.2. [23] A sequence $x = (x_k)$ is said to be λ-statistically convergent L of order α or S_λ^α-convergent to L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{\lambda_r^\alpha}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\lambda^\alpha - \lim x = L$ or $x_k \to L(S_\lambda^\alpha)$.

Definition 5.3. Let $\lambda = (\lambda_r)$ be a sequence in Λ. A sequence $x = (x_k)$ is said to be almost λ-statistically $[A, \Delta]$-convergent or $\widehat{S}_\lambda[A, \Delta]$–convergent to L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{\lambda_r}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = 0.$$

In this case we write $\widehat{S}_\lambda[A, \Delta] - \lim x = L$ or $x_k \to L(\widehat{S}_\lambda[A, \Delta])$.

Theorem 5.1. *Let $\lambda = (\lambda_r)$ be a sequence in Λ, then*

(a) *If $x_k \to L(\widehat{V}_\lambda[A, \Delta])$ then $x_k \to L(\widehat{S}_\lambda[A, \Delta])$.*
(b) *If $x \in l_\infty[A, \Delta]$ and $x_k \to L(\widehat{S}_\lambda[A, \Delta])$, then $x_k \to L(\widehat{V}_\lambda[A, \Delta])$.*
(c) *$\widehat{V}_\lambda[A, \Delta] \cap l_\infty[A, \Delta] = \widehat{S}_\lambda[A, \Delta] \cap l_\infty[A, \Delta]$, where*

$$l_\infty[A, \Delta] = \left\{x \in w : \sup_{k,n} |\Delta B_{kn}(x)| < \infty\right\}.$$

Proof. (a) Suppose that $\varepsilon > 0$ and $x_k \to L(\widehat{V}_\lambda[A, \Delta])$, then we have

$$\sum_{k \in I_r} |\Delta B_{kn}(x) - L| \geq \sum_{\substack{k \in I_r \\ |\Delta B_{kn}(x) - L| \geq \varepsilon}} |\Delta B_{kn}(x) - L|$$

$$\geq \varepsilon|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}|.$$

Therefore $x_k \to L(\widehat{S}_\lambda[A, \Delta])$.

(b) Suppose that $x \in l_\infty[A, \Delta]$ and $x_k \to L(\widehat{S}_\lambda[A, \Delta])$, i.e., for some $K > 0$, $|\Delta B_{kn}(x) - L| \leq K$ for all k and n. Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{\lambda_r}\sum_{k \in I_r} |\Delta B_{kn}(x) - L| &= \frac{1}{\lambda_r}\sum_{\substack{k \in I_r \\ |\Delta B_{kn}(x) - L| \geq \varepsilon}} |\Delta B_{kn}(x) - L| \\ &\quad + \frac{1}{\lambda_r}\sum_{\substack{k \in I_r \\ |\Delta B_{kn}(x) - L| < \varepsilon}} |\Delta B_{kn}(x) - L| \\ &\leq \frac{K}{\lambda_r}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| + \varepsilon,\end{aligned}$$

as $r \to \infty$, the right side goes to zero, which implies that $x_k \to L(\widehat{V}_\lambda[A, \Delta])$.

(c) Follows from (a) and (b). □

Definition 5.4. Let $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be almost statistically $[A, \Delta]$– convergent to L of order α or $\widehat{S}^\alpha[A, \Delta]$-convergent to L of order α if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n^\alpha}|\{k \leq n : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = 0.$$

In this case we write $\widehat{S}^\alpha[A, \Delta] - \lim x = L$ or $x_k \to L(\widehat{S}^\alpha[A, \Delta])$.

Definition 5.5. Let $\lambda = (\lambda_r)$ be a sequence in Λ, and $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be almost λ-statistically-$[A, \Delta]$-convergent to L of order α or $\widehat{S}_\lambda^\alpha[A, \Delta]$–convergent to L of order α if for every $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = 0.$$

In this case we write $\widehat{S}_\lambda^\alpha[A, \Delta] - \lim x = L$ or $x_k \to L(\widehat{S}_\lambda^\alpha[A, \Delta])$.

Theorem 5.2. *For $0 < \alpha \leq 1$, if $\widehat{S}^\alpha[A, \Delta] - \lim_k x_k = x_0$ then x_0 is unique.*

Proof. The proof of the result is easy, so omitted. □

Theorem 5.3. *Let $0 < \alpha \leq 1$ and $x = (x_k)$ and $(y = (y_k))$ be sequences of real numbers.*

(a) *If $\widehat{S}^\alpha[A, \Delta] - \lim_k x_k = x_0$ and $c \in \mathbb{C}$, then $\widehat{S}^\alpha[A, \Delta] - \lim_k(cx_k) = cx_0$.*
(b) *If $\widehat{S}^\alpha[A, \Delta] - \lim_k x_k = x_0$ and $\widehat{S}^\alpha[A, \Delta] - \lim_k y_k = y_0$, then $\widehat{S}^\alpha[A, \Delta] - \lim_k(x_k + y_k) = x_0 + y_0$.*

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\begin{aligned}&\frac{1}{n^\alpha}|\{k \leq n : |\Delta B_{kn}(cx) - cx_0| \geq \varepsilon\}| \\ &\quad = \frac{1}{n^\alpha}\left|\left\{k \leq n : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{|c|}\right\}\right|.\end{aligned}$$

(b) For every $\varepsilon > 0$. The result follows from the from the following inequality.

$$\begin{aligned}&\frac{1}{n^\alpha}|\{k \leq n : |\Delta B_{kn}(x + y) - (x_0 + y_0)| \geq \varepsilon\}| \\ &\quad \leq \frac{1}{n^\alpha}\left|\left\{k \leq n : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{2}\right\}\right| \\ &\qquad + \frac{1}{n^\alpha}\left|\left\{k \leq n : |\Delta B_{kn}(y) - y_0| \geq \frac{\varepsilon}{2}\right\}\right|\end{aligned}$$

□

Theorem 5.4. *Let $0 < \alpha \leq 1$ and $x = (x_k)$ and $(y = (y_k))$ be sequences of real numbers.*

(a) *If $\widehat{S}_\lambda^\alpha[A, \Delta] - \lim_k x_k = x_0$ and $c \in \mathbb{C}$, then $\widehat{S}_\lambda^\alpha[A, \Delta] - \lim_k(cx_k) = cx_0$.*
(b) *If $\widehat{S}_\lambda^\alpha[A, \Delta] - \lim_k x_k = x_0$ and $\widehat{S}_\lambda^\alpha[A, \Delta] - \lim_k y_k = y_0$, then $\widehat{S}_\lambda^\alpha[A, \Delta] - \lim_k(x_k + y_k) = x_0 + y_0$.*

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\begin{aligned}&\frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(cx) - cx_0| \geq \varepsilon\}| \\ &\quad = \frac{1}{\lambda_r^\alpha}\left|\left\{k \in I_r : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{|c|}\right\}\right|.\end{aligned}$$

(b) For every $\varepsilon > 0$. The result follows from the from the following inequality.

$$\frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x + y) - (x_0 + y_0)| \geq \varepsilon\}|$$

$$\leq \frac{1}{\lambda_r^\alpha}\left|\left\{k \in I_r : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\lambda_r^\alpha}\left|\left\{k \in I_r : |\Delta B_{kn}(y) - y_0| \geq \frac{\varepsilon}{2}\right\}\right|$$

□

Theorem 5.5. *If $0 < \alpha < \beta \leq 1$, then $\widehat{S}_\lambda^\alpha[A, \Delta] \subset \widehat{S}_\lambda^\beta[A, \Delta]$ and the inclusion is strict.*

Proof. The proof of the result follows form the following equality.

$$\frac{1}{\lambda_r^\beta}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = \frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}|.$$

To prove the inclusion is strict, let λ be given and we consider a sequence $x = (x_k)$ be defined by

$$\Delta B_{kn}(x_k) = \begin{cases} k, & \text{if } r - [\sqrt{\lambda_r}] + 1 \leq k \leq r; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{\lambda_r^\beta}|\{k \in I_r : |\Delta B_{kn}(x_k) - 0| \geq \varepsilon\}| = \frac{1}{\lambda_r^\beta}|\{k \in I_r : r - [\sqrt{\lambda_r}] + 1 \leq k \leq r\}| \leq \frac{\sqrt{\lambda_r}}{\lambda_r^\beta}$$

Then we have $x \in \widehat{S}_\lambda^\beta[A, \Delta]$ for $\frac{1}{2} < \beta \leq 1$ but $x \notin \widehat{S}_\lambda^\alpha[A, \Delta]$ for $0 < \alpha \leq \frac{1}{2}$. □

Corollary 5.6. *If a sequence is $\widehat{S}_\lambda^\alpha[A, \Delta]$-convergent to L then it is $\widehat{S}_\lambda[A, \Delta]$-convergent to L for $0 < \alpha \leq 1$.*

Theorem 5.7. *Let $0 < \alpha \leq 1$ and $\lambda = (\lambda_r) \in \Lambda$. Then $\widehat{S}^\alpha[A, \Delta] \subset \widehat{S}_\lambda^\alpha[A, \Delta]$ if*

$$\liminf_{r\to\infty} \frac{\lambda_r^\alpha}{r^\alpha} > 0.$$

Proof. If $x_k \to L\left(\widehat{S}^\alpha[A, \Delta]\right)$ then for every $\varepsilon > 0$ and for sufficiently large r we have

$$\frac{1}{r^\alpha}|\{k \leq r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| \geq \frac{1}{r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| \geq \frac{\lambda_r^\alpha}{r^\alpha}\frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}|.$$

Taking the limit as $r \to \infty$ and using the given condition, we get $x_k \to L\left(\widehat{S}_\lambda^\alpha[A, \Delta]\right)$. This completes the proof of the theorem. □

Corollary 5.8. *Let $0 < \alpha \leq 1$ and $\lambda = (\lambda_r) \in \Lambda$. Then $\widehat{S}_\lambda^\alpha[A, \Delta] \subset \widehat{S}[A, \Delta]$.*

Theorem 5.9. *Let $0 < \alpha \leq 1$ and $\lambda = (\lambda_r) \in \Lambda$. Then $\widehat{S}[A, \Delta] \subset \widehat{S}_\lambda^\alpha[A, \Delta]$ if and only if*

$$\liminf_{r\to\infty} \frac{\lambda_r^\alpha}{r} > 0. \tag{5.1}$$

Proof. Let the condition (5.1) holds and $x = (x_k) \in \widehat{S}[A, \Delta]$. For a given $\varepsilon > 0$ we have

$$\{k \leq r : |\Delta B_{kn}(x) - L| \geq \varepsilon\} \supset \{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}.$$

Then we have

$$\frac{1}{r}|\{k \leq r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| \geq \frac{1}{r}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = \frac{\lambda_r^\alpha}{r}\frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}|.$$

By taking limit as $r \to \infty$ and from relation (5.1) we have

$$x_k \to L\left(\widehat{S}[A, \Delta]\right) \Rightarrow x_k \to L\left(\widehat{S}_\lambda^\alpha[A, \Delta]\right).$$

Next we suppose that

$$\liminf_{r\to\infty} \frac{\lambda_r^\alpha}{r} = 0.$$

Then we can choose a subsequence (r_i) such that $\frac{\lambda_{r_i}^\alpha}{r_i} < \frac{1}{i}$. Define a sequence $x = (x_k)$ as follows:

$$\Delta B_{kn}(x_k) = \begin{cases} 1, & \text{if } k \in I_{r_i}; \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly $x = (x_k) \in \widehat{S}[A, \Delta]$ but $x = (x_k) \notin \widehat{S}_\lambda[A, \Delta]$. Since $\widehat{S}_\lambda^\alpha[A, \Delta] \subset \widehat{S}_\lambda[A, \Delta]$, we have $x = (x_k) \notin \widehat{S}_\lambda^\alpha[A, \Delta]$, which is a contradiction. Hence the relation (5.1) holds. □

Theorem 5.10. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If*

$$\liminf_{r\to\infty} \frac{\lambda_r^\alpha}{\mu_r^\beta}, \tag{5.2}$$

then $\widehat{S}_\mu^\beta[A, \Delta] \subseteq \widehat{S}_\lambda^\alpha[A, \Delta]$.

Proof. Suppose that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and the condition (5.2) hold. Then $I_r \subset J_r$ and so that for $\varepsilon > 0$ we can write

$$\{k \in J_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\} \supset \{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}.$$

Then we have

$$\frac{1}{\mu_r^\beta}|\{k \in J_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| \geq \frac{\lambda_r^\alpha}{\mu_r^\beta}\frac{1}{\lambda_r^\alpha}|\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}|,$$

for all $r \in \mathbb{N}$, where $J_r = [r - \mu_r + 1, r]$. Taking limit $r \to \infty$ in the last inequality and using (5.2), we have $\widehat{S}_\mu^\beta[A, \Delta] \subseteq \widehat{S}_\lambda^\alpha[A, \Delta]$. □

Corollary 5.11. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$. If (5.2) holds, then*

(a) $\widehat{S}_\mu^\alpha[A, \Delta] \subseteq \widehat{S}_\lambda^\alpha[A, \Delta]$ *for $0 < \alpha \leq 1$,*
(b) $\widehat{S}_\mu[A, \Delta] \subseteq \widehat{S}_\lambda^\alpha[A, \Delta]$ *for $0 < \alpha \leq 1$,*

(c) $\widehat{S}_{\mu}[A,\Delta] \subseteq \widehat{S}_{\lambda}[A,\Delta]$.

Theorem 5.12. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\leq\beta\leq 1$. If*

$$\lim_{r\to\infty}\frac{\mu_r}{\lambda_r^{\beta}}=1, \tag{5.3}$$

then $\widehat{S}_{\lambda}^{\alpha}[A,\Delta] \subseteq \widehat{S}_{\mu}^{\beta}[A,\Delta]$.

Proof. Let $\widehat{S}_{\lambda}^{\alpha}[A,\Delta]-\lim x=L$ and (5.3) be satisfied. Since $I_r\subset J_r$, for $\varepsilon>0$ we can write

$$\begin{aligned}
&\frac{1}{\mu_r^{\beta}}|\{k\in J_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|\\
&\quad=\frac{1}{\mu_r^{\beta}}|\{r-\mu_r+1\leq k\leq r-\lambda_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|\\
&\qquad+\frac{1}{\mu_r^{\beta}}|\{k\in I_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|\\
&\quad\leq\frac{\mu_r-\lambda_r}{\mu_r^{\beta}}+\frac{1}{\mu_r^{\beta}}|\{k\in I_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|\\
&\quad\leq\frac{\mu_r-\lambda_r^{\beta}}{\lambda_r^{\beta}}+\frac{1}{\mu_r^{\beta}}|\{k\in I_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|\\
&\quad\leq\left(\frac{\mu_r}{\lambda_r^{\beta}}-1\right)+\frac{\lambda_r^{\alpha}}{\mu_r^{\beta}}\frac{1}{\lambda_r^{\alpha}}|\{k\in I_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|.
\end{aligned}$$

Using the relation (5.3) and $\widehat{S}_{\lambda}^{\alpha}[A,\Delta]-\lim x=L$ the right-hand side of the above inequality tends to zero as $r\to\infty$. This implies that $\widehat{S}_{\lambda}^{\alpha}[A,\Delta]\subseteq\widehat{S}_{\mu}^{\beta}[A,\Delta]$. □

Corollary 5.13. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\leq\mu_r$ for all $r\in\mathbb{N}$. If (5.3) holds, then*

(a) $\widehat{S}_{\lambda}^{\alpha}[A,\Delta]\subseteq\widehat{S}_{\mu}^{\alpha}[A,\Delta]$ *for* $0<\alpha\leq 1$,
(b) $\widehat{S}_{\lambda}[A,\Delta]\subseteq\widehat{S}_{\mu}^{\alpha}[A,\Delta]$ *for* $0<\alpha\leq 1$,
(c) $\widehat{S}_{\lambda}[A,\Delta]\subseteq\widehat{S}_{\mu}[A,\Delta]$.

Definition 5.6. Let M be an Orlicz function, $p=(p_k)$ be a sequence of strictly positive real numbers, $\alpha\in(0,1]$, $\lambda=(\lambda_r)$ be a sequence of positive reals, and for $\rho>0$, now we define

$$\begin{aligned}
&\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]\\
&\quad=\left\{x\in w:\ \lim_{r\to\infty}\frac{1}{\lambda_r^{\alpha}}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}=0,\right.\\
&\qquad\left.\text{for some } L,\ \text{ uniformly on } n\right\}.
\end{aligned}$$

If $M(x)=x$ and $p_k=p$ for all $k\in\mathbb{N}$ then we shall write $\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]=\widehat{V}_{\lambda}^{\alpha}[A,\Delta](p)$ and if $M(x)=x$ then we shall write $\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]=\widehat{V}_{\lambda}^{\alpha}[A,\Delta,p]$.

Theorem 5.14. *Let (p_k) be a bounded and $0<\inf_k p_k\leq p_k\leq \sup_k p_k=H<\infty$. Let $0<\alpha\leq\beta\leq 1$, M be an Orlicz function and $\lambda=(\lambda_r)$ be a sequence of positive reals, then $\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]\subset\widehat{S}_{\lambda}^{\beta}[A,\Delta]$.*

Proof. Let $x=(x_k)\in\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]$. Let $\varepsilon>0$ be given. As $h_r^{\alpha}\leq h_r^{\beta}$ for each r we can write

$$\begin{aligned}
&\frac{1}{\lambda_r^{\alpha}}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}\\
&=\frac{1}{\lambda_r^{\alpha}}\left[\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|\geq\varepsilon}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}+\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|<\varepsilon}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}\right]\\
&\geq\frac{1}{\lambda_r^{\beta}}\left[\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|\geq\varepsilon}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}+\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|<\varepsilon}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}\right]\\
&\geq\frac{1}{\lambda_r^{\beta}}\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|\geq\varepsilon}}\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\\
&\geq\frac{1}{\lambda_r^{\beta}}\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|\geq\varepsilon}}\min\left([M(\varepsilon_1)]^{h},[M(\varepsilon_1)]^{H}\right),\ \varepsilon_1=\frac{\varepsilon}{\rho}\\
&\geq\frac{1}{\lambda_r^{\beta}}|\{k\in I_r:|\Delta B_{kn}(x)-L|\geq\varepsilon\}|\\
&\quad\min\left([M(\varepsilon_1)]^{h},[M(\varepsilon_1)]^{H}\right).
\end{aligned}$$

From the above inequality we have $(x_k)\in\widehat{S}_{\lambda}^{\beta}[A,\Delta]$. □

Corollary 5.15. *Let $0<\alpha\leq 1$, M be an Orlicz function and $\lambda=(\lambda_r)$ be an element of Λ, then $\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]\subset\widehat{S}_{\lambda}^{\alpha}[A,\Delta]$.*

Theorem 5.16. *Let M be an Orlicz function, $x=(x_k)$ be a sequence in $l_\infty[A,\Delta]$, and $\lambda=(\lambda_r)$ be an element of Λ. If $\lim_{r\to\infty}\frac{\lambda_r}{\lambda_r^{\alpha}}=1$, then $\widehat{S}_{\lambda}^{\alpha}[A,\Delta]\subset\widehat{V}_{\lambda}^{\alpha}[A,M,\Delta,p]$.*

Proof. Suppose that $x=(x_k)$ is a sequence in $l_\infty[A,\Delta]$ and $\widehat{S}^{\alpha}[A,\Delta]-\lim_k x_k=L$. As $x=(x_k)\in l_\infty[A,\Delta]$ there exists $K>0$ such that $|\Delta B_{kn}(x)|\leq K$ for all k and n. For given $\varepsilon>0$ we have

$$\begin{aligned}
&\frac{1}{\lambda_r^{\alpha}}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}\\
&=\frac{1}{\lambda_r^{\alpha}}\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|\geq\varepsilon}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}+\frac{1}{\lambda_r^{\alpha}}\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|<\varepsilon}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k}\\
&\leq\frac{1}{\lambda_r^{\alpha}}\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|\geq\varepsilon}}\max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^{h},\left[M\left(\frac{K}{\rho}\right)\right]^{H}\right\}+\frac{1}{\lambda_r^{\alpha}}\sum_{\substack{k\in I_r\\|\Delta B_{kn}(x)-L|<\varepsilon}}\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\\
&\leq\max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^{h},\left[M\left(\frac{K}{\rho}\right)\right]^{H}\right\}\frac{1}{\lambda_r^{\alpha}}||\Delta B_{kn}(x)-L|\geq\varepsilon|
\end{aligned}$$

$$+\frac{\lambda_r}{\lambda_r^\alpha}\max\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h,\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}.$$

Therefore we have $(x_k)\in\widehat{V}_\lambda^\alpha[A,M,\Delta,p]$. □

Theorem 5.17. *Let $\lambda=(\lambda_r)\in\Lambda$, $0<\alpha\le\beta\le1$, p be a positive real number, then $\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{V}_\lambda^\beta[A,\Delta](p)$.*

Proof. The proof is easy, so omitted. □

Corollary 5.18. *Let $\lambda=(\lambda_r)\in\Lambda$ and p be a positive real number, then $\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{V}_\lambda[A,\Delta](p)$.*

Theorem 5.19. *Let $\lambda=(\lambda_r)\in\Lambda$, $0<\alpha\le\beta\le1$ and p be a positive real number, then $\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{S}_\lambda^\beta[A,\Delta]$.*

Proof. Let $x=(x_k)\in\widehat{V}_\lambda^\alpha[A,\Delta](p)$ and for $\varepsilon>0$ we have

$$\begin{aligned}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p&=\sum_{\substack{k\in I_r\\ |\Delta B_{kn}(x)-L|\ge\varepsilon}}|\Delta B_{kn}(x)-L|^p\\&\quad+\sum_{\substack{k\in I_r\\ |\Delta B_{kn}(x)-L|<\varepsilon}}|\Delta B_{kn}(x)-L|^p\\&\ge\sum_{\substack{k\in I_r\\ |\Delta B_{kn}(x)-L|\ge\varepsilon}}|\Delta B_{kn}(x)-L|^p\\&\ge|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}|.\varepsilon^p.\end{aligned}$$

Therefore we have

$$\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p\ge\frac{1}{\lambda_r^\beta}|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}|.\varepsilon^p.$$

The last inequality implies that $x=(x_k)\in\widehat{S}_\lambda^\beta[A,\Delta]$ if $x=(x_k)\in\widehat{V}_\lambda^\alpha[A,\Delta](p)$. This completes the proof of the theorem. □

Theorem 5.20. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\le\mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\le\beta\le1$. If (5.2) holds, then $\widehat{V}_\mu^\beta[A,\Delta](p)\subseteq\widehat{V}_\lambda^\alpha[A,\Delta](p)$*

Proof. The proof is easy, so omitted. □

Corollary 5.21. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\le\mu_r$ for all $r\in\mathbb{N}$. If (5.2) holds, then*

(a) $\widehat{V}_\mu^\alpha[A,\Delta](p)\subseteq\widehat{V}_\lambda^\alpha[A,\Delta](p)$ *for* $0<\alpha\le1$,
(b) $\widehat{V}_\mu[A,\Delta](p)\subseteq\widehat{V}_\lambda^\alpha[A,\Delta](p)$ *for* $0<\alpha\le1$,
(c) $\widehat{V}_\mu[A,\Delta](p)\subseteq\widehat{V}_\lambda[A,\Delta](p)$.

Theorem 5.22. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\le\mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\le\beta\le1$. If (5.2) holds, then $\widehat{V}_\mu^\beta[A,\Delta](p)\subseteq\widehat{S}_\lambda^\alpha[A,\Delta]$.*

Proof. Let $x=(x_k)\in\widehat{V}_\mu^\beta[A,\Delta](p)$. Then for $\varepsilon>0$ we have

$$\begin{aligned}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p&=\sum_{\substack{k\in I_r\\ |\Delta B_{kn}(x)-L|\ge\varepsilon}}|\Delta B_{kn}(x)-L|^p\\&\quad+\sum_{\substack{k\in I_r\\ |\Delta B_{kn}(x)-L|<\varepsilon}}|\Delta B_{kn}(x)-L|^p\\&\ge\sum_{\substack{k\in I_r\\ |\Delta B_{kn}(x)-L|\ge\varepsilon}}|\Delta B_{kn}(x)-L|^p\end{aligned}$$

Therefore we have

$$\begin{aligned}&\frac{1}{\mu_r^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p\\&\quad\ge\frac{\lambda_r^\alpha}{\mu_r^\beta}\frac{1}{\lambda_r^\alpha}|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}|.\varepsilon^p,\end{aligned}$$

since (5.2) holds and $x=(x_k)\in\widehat{V}_\mu^\beta[A,\Delta](p)$. The last inequality implies that $x=(x_k)\in\widehat{S}_\lambda^\alpha[A,\Delta]$. This completes the proof of the theorem. □

Corollary 5.23. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\le\mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\le1$. If (5.2) holds, then*

(a) $\widehat{V}_\mu^\alpha[A,\Delta](p)\subseteq\widehat{S}_\lambda^\alpha[A,\Delta]$,
(b) $\widehat{V}_\mu[A,\Delta](p)\subseteq\widehat{S}_\lambda^\alpha[A,\Delta]$,
(c) $\widehat{V}_\mu[A,\Delta](p)\subseteq\widehat{S}_\lambda[A,\Delta]$,

Theorem 5.24. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\le\mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\le\beta\le1$. If (5.3) holds, then $\ell_\infty[A,\Delta]\cap\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{V}_\mu^\beta[A,\Delta](p)$.*

Proof. Let $x=(x_k)\in\ell_\infty[A,\Delta]\cap\widehat{V}_\lambda^\alpha[A,\Delta](p)$ and suppose that (5.3) holds. Since $(x_k)\in\ell_\infty[A,\Delta]$, there exists $K>0$ such that $|\Delta B_{kn}(x)|\le K$ for all k and n. Since $\lambda_r\le\mu_r$ and $I_r\subset J_r$ for all $r\in\mathbb{N}$ we can write

$$\begin{aligned}\frac{1}{\mu_r^\beta}\sum_{k\in J_r}|\Delta B_{kn}(x)-L|^p&=\frac{1}{\mu_r^\beta}\sum_{k\in J_r-I_r}|\Delta B_{kn}(x)-L|^p\\&\quad+\frac{1}{\mu_r^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p\\&\le\left(\frac{\mu_r-\lambda_r}{\mu_r^\beta}\right)K^p+\frac{1}{\mu_r^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p\\&\le\left(\frac{\mu_r-\lambda_r^\beta}{\mu_r^\beta}\right)K^p+\frac{1}{\mu_r^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p\\&\le\left(\frac{\mu_r-\lambda_r^\beta}{\lambda_r^\beta}\right)K^p+\frac{\lambda_r^\alpha}{\mu_r^\beta}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p\\&\le\left(\frac{\mu_r}{\lambda_r^\beta}-1\right)K^p+\frac{\lambda_r^\alpha}{\mu_r^\beta}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)-L|^p.\end{aligned}$$

This implies that $x=(x_k)\in\widehat{V}_\mu^\beta[A,\Delta](p)$.

Hence $\ell_\infty[A,\Delta]\cap\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{V}_\mu^\beta[A,\Delta](p)$. □

Corollary 5.25. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\le\mu_r$ for all $r\in\mathbb{N}$. If (5.3) holds, then*

(a) $\ell_\infty[A,\Delta]\cap\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{V}_\mu^\alpha[A,\Delta](p)$ *for* $0<\alpha\le1$,
(b) $\ell_\infty[A,\Delta]\cap\widehat{V}_\lambda^\alpha[A,\Delta](p)\subseteq\widehat{V}_\mu[A,\Delta](p)$ *for* $0<\alpha\le1$,
(c) $\ell_\infty[A,\Delta]\cap\widehat{V}_\lambda[A,\Delta](p)\subseteq\widehat{V}_\mu[A,\Delta](p)$.

Theorem 5.26. *Let M be an Orlicz function and if $\inf_k p_k>0$, then limit of any sequence $x=(x_k)$ in $\widehat{V}_\lambda^\alpha[A,M,\Delta,p]$ is unique.*

Proof. Let $\lim_k p_k=s>0$. Suppose that $(x_k)\to l_1(\widehat{V}_\lambda^\alpha[A,M,\Delta,p])$ and $(x_k)\to l_2(\widehat{V}_\lambda^\alpha[A,M,\Delta,p])$. Then there exist $\rho_1>0$ and $\rho_2>0$ such that

$$\lim_{r\to\infty}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)-l_1|}{\rho}\right)\right]^{p_k}=0,\ \text{uniformly on } n$$

and

$$\lim_{r\to\infty}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)-l_2|}{\rho}\right)\right]^{p_k}=0,\quad \text{uniformly on } n.$$

Let $\rho=\max\{2\rho_1,2\rho_2\}$. As M is nondecreasing and convex, we have

$$\begin{aligned}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{|l_1-l_2|}{\rho}\right)\right]^{p_k}&\\ \leq \frac{D}{\lambda_r^\alpha}\sum_{k\in I_r}\frac{1}{2^{p_k}}&\left(\left[M\left(\frac{|\Delta B_{kn}(x)-l_1|}{\rho}\right)\right]^{p_k}\right.\\ &\left.+\left[M\left(\frac{|\Delta B_{kn}(x)-l_2|}{\rho}\right)\right]^{p_k}\right)\\ \frac{D}{\lambda_r^\alpha}\sum_{k\in I_r}&\left(\left[M\left(\frac{|\Delta B_{kn}(x)-l_1|}{\rho}\right)\right]^{p_k}\right.\\ &\left.+\frac{D}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)-l_2|}{\rho}\right)\right]^{p_k}\right)\to 0 \text{ as } r\to\infty,\end{aligned}$$

where $\sup_k p_k=H$ and $D=\max(1,2^{H-1})$. Therefore we get

$$\lim_{r\to\infty}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{|l_1-l_2|}{\rho}\right)\right]^{p_k}=0.$$

As $\lim_k p_k=s$, we have

$$\lim_{k\to\infty}\left[M\left(\frac{|l_1-l_2|}{\rho}\right)\right]^{p_k}=\left[M\left(\frac{|l_1-l_2|}{\rho}\right)\right]^{s}$$

and so $l_1=l_2$. Hence the limit is unique. □

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